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CONDITIONS ON SURFACES OF DISCONTINUITY IN A RIGID-PLASTIC ANALYSIS*

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Relationships are established on the surfaces of discontinuity for a rigid plastic analysis of inhomogeneous bodies, particularly bodies with piecewise-continuous properties. They are derived as necessary conditions for the static and dynamic load coefficients to be equal. In the special case of homogeneous bodies they are identical with the well-known Hill conditions; the necessity of the latter is thereby established. The connection of the different formulations of extremal problems of limit load theory is discussed in deriving the relationships. The mechanical meaning of the relationships obtained and certain of their properties are examined.

1. Formulation of the problem. Conditions on surfaces of discontinuity in a rigid plastic analysis are examined in this paper from the viewpoint of limit load theory.

The problem of the limit load theory. Let a body occupy a domain Ω and be subjected to mass forces with density $\mathbf{f}$ and a surface load with density $\mathbf{q}$ applied to a part S_q of the body surface. There is a set C_x of allowable stresses for each point x of the body. The stress field that is an inner point of the set of allowable stress fields is called safe. The main question of limit load theory is to clarify whether or not it is possible to equilibrate a given load $\mathbf{l} = (\mathbf{f}, \mathbf{q})$ and a safe stress field.

The static extremal problem. A field σ of allowable stresses is called statically allowable for a load $m\mathbf{l}$, $m \geqslant 0$, $\mathbf{l} = (\mathbf{f}, \mathbf{q})$ if it equilibrates this load; in this case the number $m_s(\sigma) = m$ is called the static coefficient of the load $\mathbf{l}$. The exact upper bound $\alpha_l = \sup m_s(\sigma)$ is called the static limit coefficient of the load $\mathbf{l}$. The load $m\mathbf{l}$ can be equilibrated to a safe stress field for $0 \leqslant m < \alpha_l$ and it is impossible to equilibrate thus for $m \geqslant \alpha_l$ /1/. Therefore, to answer the fundamental question of limit load theory it is necessary to find or estimate the quantity α_l, that is called the safety factor of the load $\mathbf{l}$ also in connection with the assertion presented.

Kinematic extremal problems. Kinematic extremal problems that are formulated in the following manner play an important part in finding the load safety factor. A dissipation function /2-6/ is associated with a set of allowable stresses C_x ($\mathbf{e}$ is a symmetric tensor of the second rank)

$$d(x; \mathbf{e}) = d_x(\mathbf{e}) = \sup\{\sigma_* \cdot \mathbf{e} \colon \sigma_* \in C_x\} \tag{1.1}$$

The strain rate $\mathbf{e}(\mathbf{u})$ and dissipation

$$\int_\Omega d_x(\mathbf{e}(\mathbf{u}))\, dx$$

*Prikl. Matem. Mekhan., 53, 3, 506-517, 1989 field correspond to a fairly regular velocity field **u**.

The velocity field **u** is called kinematically allowable if it vanishes on a part S_v of the body boundary complementing the surface S_q.

Called the kinematic coefficient $m_k(\mathbf{u})$ of the load **l** for a kinematically allowable velocity field, **u** is the ratio of the dissipation corresponding to this field to the power of the work by the given load **l** on it (if this power is positive)

$$m_k(\mathbf{u}) = \int\limits_{\Omega} d_x(\mathbf{e}(\mathbf{u}))\,dx\ \Big(\int\limits_{\Omega} \mathbf{fu}\,dx + \int\limits_{S_q} \mathbf{qu}\,ds\Big)^{-1}$$

An analogous definition is retained if the set of kinematically allowable velocity fields is expanded to a certain space of generalized functions and the dissipation and power of the work of the load are, respectively, continued into this space /4, 7, 8/. The kinematic load factor for the generalized velocity field **v** is denoted by $M_k(\mathbf{v})$.

The quantities

$$B_l = \inf m_k(\mathbf{u}),\ \beta_l = \inf M_k(\mathbf{v})$$

are called kinematic limit factors, where the extremum is sought in the first problem for smooth, and in the second for generalized kinematically allowable velocity fields.

Assertion 1 /1, 2, 4/. Every static coefficient of the load **l** does not exceed its kinematic factor

$$m_s(\sigma) \leqslant m_k(\mathbf{u}),\ m_s(\sigma) \leqslant M_k(\mathbf{v})$$

This means that it is sufficient to construct a stress field σ and a velocity field $\mathbf{u}(\mathbf{v})$ for which equality would be achieved in the previous relationship in order to find the safety factor of the load α_l. Indeed, then by the Assertion 1 the common value of the static and kinematic coefficient equals α_l.

Relation to the problem of rigid plastic analysis. Such stress and velocity fields can be sought by relying on the following assertion.

Assertion 2 /1, 4/. The stress σ and velocity $\mathbf{u} \neq 0\ (\mathbf{v} \neq 0)$ fields are a strong smooth (respectively, weak) solution of the problem of rigid plastic analysis for the load $m\mathbf{l},\ m > 0$, if and only if the static $m_s(\sigma) = m$ and kinematic $m_k(\mathbf{u})\ (M_k(\mathbf{v}))$ factors of the load **l** are identical.

The rigid plastic analysis problem (RPAP) is understood here to be the following. A rigid ideally plastic body is considered whose flow surface is the boundaries of the set C_x (C_x is a convex set of allowable stresses yielding the body properties in the original formulation of the problem of limit load theory). A statically allowable stress field σ for **l** and a kinematically allowable smooth velocity field **u** satisfying the associated law is called a strong smooth solution of the RPAP for the load **l**.

Satisfaction of the (normal, gradient) law associated with the set C_x for the velocity field **u** and stress field σ means that the strain rates **e** corresponding to **u** are directed along the "external normal" to the set C_x

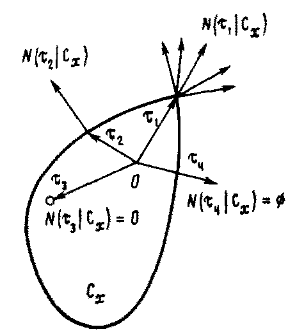

Fig.1

$$\mathbf{e}(x) \in N(\sigma(x) \mid C_x) \tag{1.2}$$

Here $N(\tau \mid C_x)$ is the normal cone of the set C_x at the point τ /9/ (Fig.1)

$$N(\tau \mid C_x) = \begin{cases} \{\varepsilon : (\tau - \tau_*)\cdot\varepsilon \geqslant 0 \quad \forall \tau_* \in C_x\}, & \tau \in C_x \\ \varnothing, \quad \tau \notin C_x \end{cases}$$

The weak solution of this problem is defined analogously (the smooth velocity fields are replaced by generalized fields, and the associated law is weakened /4/).

Thus the solution of the RPAP is a method for finding the load safety factor. Assertion 2 clarifies the role of the associated law in the limit load theory: it does not occur as a physical governing relationship but as the necessary and sufficient condition for agreement between the static and kinematic load factors and in this sense is finally given a foundation.

The idea for deriving conditions on the surfaces of discontinuity. Conditions on surfaces of discontinuity can be given a foundation similar to the associated law. From the viewpoint of limit load theory they should satisfy the following requirement. The discontinuous RPAP solutions determined by using these conditions should result in the value of the load safety factor (as the smooth solution when it exists).

For homogeneous bodies the conditions mentioned in /10/ (the jump in the function f in the direction of the normal $\mathbf{v}$ to the surface of discontinuity is denoted by $[f]$; f^+, f^- are limit values of the function f from two sides of the discontinuity surface

$$(\varepsilon_{\nu}(a))_{ij} = (a_i\nu_j + a_j\nu_i)/2$$
$$[\sigma_{ij}]\,\nu_j = 0,\quad \varepsilon_{\nu}([\mathbf{u}]) \in N(\sigma^+ \mid C_x),\quad \varepsilon_{\nu}([\mathbf{u}]) \in N(\sigma^- \mid C_x) \tag{1.3}$$

Different conditions from these relations have been proposed for the discontinuity surfaces in homogeneous plastic bodies. It follows from the sequel that only conditions (1.3) satisfy the requirement mentioned.

The purpose of this paper is to derive conditions on discontinuity surfaces for inhomogeneous bodies that would satisfy the formulated requirement (acceptability for finding the load safety factor).

Weak RPAP solutions σ, $\mathbf{v}$ on which the equality $m_s(\sigma) = M_k(\mathbf{v})$ is known to be achieved, are considered for deriving such relationships. When such a solution is reduced to discontinuous piecewise-smooth stress and velocity fields, the conditions on the discontinuity surfaces of these fields can be found from the determination of the weak solution.

By such means relationships were obtained on discontinuity surfaces for elastic-plastic bodies described by the Prandtl-Reuss equations /11/ and the relationships (1.3) /12/.

An integral relation (Sect.2) and later relationships on the discontinuity surfaces (Sect.3) are derived from an examination of the weak solutions. The simple form and mechanical meaning of these relations are established in Sect.4 and some of their properties in Sect.5.

Let us note certain assumptions and notation utilized later.

The domain Ω is located in Euclidean space. Its dimensionality affects only certain constants, for instance, when compiling the global part of a tensor or when using the imbedding theorem. The values of such constants are later mentioned for the three-dimensional case.

Let Sym denote the space of symmetric tensors of second rank, s^d and s^s are, respectively, the deviator and global components of the tensor s from Sym, and Sym^d is the subspace of tensor-deviators (deviator plane). The notation $\mathbf{a}\cdot\mathbf{b} = a_{ij}b^{ij}$, $|\mathbf{a}| = (\mathbf{a}\cdot\mathbf{a})^{1/2}$ is used for the second-rank tensor $\mathbf{a}$, $\mathbf{b}$. The metric tensor is denoted by $\mathbf{g}$.

Further assertions are formulated for the sets C_x in the form of cylinders in the space Sym with axis directed along the tensor $\mathbf{g}$. They are also valid for bounded sets C_x. The dependence of C_x on x is considered measurable /9/. This condition is satisfied in all cases of interest from the mechanics viewpoint.

A section of the cylinder C_x by the deviator plane Sym^d is denoted by C_x^d. It is assumed that the sets C_x^d are bounded in a set and contain the identical neighbourhood of zero of the space Sym^d. In this case the equality $\alpha_l = \beta_l$ /13/ holds while the equality $\beta_l = B_l$ /13, 14/ also holds for a certain smoothness of the domain boundary.

The space of linear continuous functionals in the space X is denoted by X'. The value of the functional f from X' on an element x from X is denoted by $\langle x, f\rangle = f(x)$.

If A is a linear operator from the space X into the space Y, then A^T is the adjoint operator (from Y' into X') while A^{TT} is the second adjoint operator (from X'' into Y'').

2. The integral relation. Let us examine the condition for the static coefficient $m_s(\sigma)$ and the kinematic coefficient $M_k(\mathbf{v})$ of the load $\mathbf{l} = (\mathbf{f}, \mathbf{q})$ to be equal. Their equality is assured for the RPAP solution σ, $\mathbf{v}$ in the weak formulation of Assertion 2 (Sect. 1).

See /15/ for details and citations of publications in which this problem has been studied. We will present just the main definitions.

We consider the space of the stress fields

$$\mathbf{S} = \mathbf{S}(\Omega) = \{\sigma : \sigma_{ij} = \sigma_{ji},\quad \sigma_{ij}{}^d \in L_\infty(\Omega),\ \sigma_{kk} \in L_2(\Omega)\}$$
$$\|\sigma\|_{\mathbf{S}}{}^2 = \|\,|\sigma^d|\,\|_{L_\infty}^2 + \|{}^1/_3\,\sigma_{kk}\|_{L_2}^2$$

The conditions for equilibrium of the stress field τ from $\mathbf{S}$ with the load $\mathbf{l} = (\mathbf{f}, \mathbf{q})$ are given by the principle of virtual velocities

$$\int_\Omega \tau\cdot\mathrm{Def}_0\,\mathbf{u}\,dx = \int_\Omega \mathbf{f}\mathbf{u}\,dx + \int_{S_q} \mathbf{q}\mathbf{u}\,ds,\quad \forall\mathbf{u} \in \mathbf{U} \tag{2.1}$$
$$\mathbf{U} = \mathbf{U}(\Omega, S_v) = \{\mathbf{u} \in \mathbf{C}^\infty(\bar\Omega) : \rho(\mathrm{supp}\,\mathbf{u}, S_v) > 0\}$$
$$(\mathrm{Def}_0\,\mathbf{u})_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right)$$

Let s_l be some stress field from the space S equilibrating the load l while Σ is the set of stress fields equilibrating the load $l = 0\ (f = 0, q = 0)$. Then $\Sigma + ms_l$ is the set of stress fields equilibrating the load l.

We also consider the space of velocity fields V_0 and the space $F_{(1)}$

$$V_0 = V_0(\Omega, S_v) = \left\{u \in L_1(\Omega) : \mathrm{Def}_0\, u \in L_1(\Omega),\ \frac{\partial u_k}{\partial x^k} \in L_2(\Omega),\ u|_{S_v} = 0\right\}$$
$$\|u\|^2_{V_0} = \|u\|^2_{L_1} + \|\,|(\mathrm{Def}_0 u)^d|\,\|^2_{L_1} + \left\|\frac{1}{3}\frac{\partial u_k}{\partial x^k}\right\|^2_{L_2}$$
$$F_{(1)} = F_{(1)}(\Omega) = \{\varepsilon : \varepsilon_{ij} = \varepsilon_{ji} \in L_1(\Omega),\ \varepsilon_{kk} \in L_2(\Omega)\}$$
$$\|\varepsilon\|^2_{F_{(1)}} = \|\,|\varepsilon^d|\,\|^2_{L_1} + \|{}^1/_3\varepsilon_{kk}\|^2_{L_2}$$

Obviously $F'_{(1)} = S$ and Def_0 is a continuous operator acting from V_0 into $F_{(1)}$. Let Ω and S_v be such that $\|u\|_{L_1} \leqslant c\|\mathrm{Def}_0\, u\|_{L_1}$ for any u from V_0 (see /16/ for conditions sufficient for this). Then the operator $\mathrm{Def} = (\mathrm{Def}_0)^{TT}$ is a continuous continuation of Def_0 into the space V_0''. It maps the space of generalized velocity fields V_0'' one-to-one on the set E of kinematically allowable generalized strain rate fields /8/

$$E = \Sigma^\circ = \{\varepsilon \in S' : \langle \tau, \varepsilon\rangle = 0\ \ \forall \tau \in \Sigma\}$$

The set C of allowable stress fields, the dissipation D, and the weak form of the associated law are defined by the relationships

$$C = \{\sigma \in S : \sigma(x) \in C_x \quad \text{for o.v.} \quad x \in \Omega\}$$
$$D(e) = \sup\{\langle \sigma_*, e\rangle : \sigma_* \in C\}, \quad e \in S'$$
$$\sigma \in C, \quad D(e) = \langle \sigma, e\rangle, \quad e \in S'$$

The stress field $\sigma \in S$ and the velocity field $v \in V_0''$ satisfying the relations

$$\sigma \in \Sigma + \alpha_l s_l, \quad \sigma \in C, \quad D(e) = \langle \sigma, e\rangle, \quad e = \mathrm{Def}\, v \tag{2.2}$$

is called the weak RPAP solution for the load $\alpha_l l$.

The kinematic coefficient for any genealized velocity field $v \in V_0''$ satisfying the condition of positivity of the power $\langle s_l, \mathrm{Def}\, v\rangle > 0$, is determined by the relationship

$$M_k(v) = D(\mathrm{Def}\, v)/\langle s_l, \mathrm{Def}\, v\rangle$$

and the static and kinematic limit coefficients as the extrema

$$\alpha_l = \sup\{m \geqslant 0 : (\Sigma + ms_l) \cap C \neq \varnothing\}$$
$$\beta_l = \inf\{D(\varepsilon)/\langle s_l, \varepsilon\rangle : \varepsilon \in E,\ \langle s_l, \varepsilon\rangle > 0\} \tag{2.3}$$

Let the equality $m_s(\sigma) = M_k(v)$ be satisfied. Then the pair σ, v is a solution of problem (2.2) $\alpha_l = m_s(\sigma) = M_k(v)$ and the relations (2.2) can be utilized to derive conditions on surfaces of discontinuity

Contraction of the generalized velocity field. There is no natural embedding of piecewise-smooth functions undergoing discontinuity on a certain surface in the space V_0''. Consequently, instead of $v \in V_0''$, $e = \mathrm{Def}\, v \in S'$ their contractions $v^{(r)} = v|_\Lambda$, $e^{(r)} = e|_{S^{(r)}}$ in a certain space Λ, $S^{(r)}$; are considered. The contractions can already be discontinuous functions.

Let Λ and $S^{(r)}$ be Banach spaces continuously embedded in V_0' and S, respectively (Assumption 1). Then the contractions $v^{(r)}, e^{(r)}$ are linear continuous functionals in Λ, $S^{(r)}$. Moreover, if $\mathrm{Def}_0{}^T\tau$ for any field τ from $S^{(r)}$ belongs to the space Λ (Assumption 2), then

$$e^{(r)} = \mathrm{Def}^{(r)} v^{(r)}, \quad \mathrm{Def}^{(r)} = (\mathrm{Def}_0{}^T|_{S^{(r)}})^T : \Lambda' \to S' \tag{2.4}$$

follows from the relationships $e = \mathrm{Def}\, v$, $v \in V_0''$.

Let us indicate the relations which the field σ, $v^{(r)}$ satisfy if $m_s(\sigma) = M_k(v)$ or, equivalently, σ, $e = \mathrm{Def}\, v$ are extremals of the Problems (2.3) or, finally σ, v is a solution of the Problem (2.2).

Connection of different formulations of limit load theory extremal problems. Let us introduce the sets $\Sigma^{(r)}$, $C^{(r)}$, $E^{(r)}$ and the functional $D^{(r)}$ be replacing the space S by the space $S^{(r)}$ in the definitions Σ, C, E, D. Let the load l be equilibrated by the space field $s_l \in S^{(r)}$. Let us examine the problem analogous to (2.3) of finding the extrema $\alpha_l^{(r)}$, $\beta_l^{(r)}$ and the extremals on which they are achieved

$$\alpha_l^{(r)} = \sup\{m \geqslant 0 : (\Sigma^{(r)} + ms_l) \cap C^{(r)} \neq \varnothing\} \tag{2.5}$$
$$\beta_l^{(r)} = \inf\{D^{(r)}(\varepsilon)/\langle s_l, \varepsilon\rangle : \varepsilon \in E^{(r)},\ \langle s_l, \varepsilon\rangle > 0\}$$

These problems are formally duals; consequently $\alpha_l^{(r)} \leqslant \beta_l^{(r)}$ /9/.

We assume that elements of the space Λ can be considered as loads $(\mathbf{f}, \mathbf{q})$ for which the power is determined - the right-hand sides of (2.1). For any load from Λ let the stress field equilibrating it also belongs to the space $S^{(r)}$ (Assumption 3).

Then the first problem of(2.5) will obviously agree with the first problem of (2.3) and $\alpha_l^{(r)} = \alpha_l$.

Let the set U be compact in the space $\mathbf{V}_0$, for example, the boundary $\partial\Omega$ consists of several connectedness components, where the spacing between its parts S_v and S_q is positive. It then follows from the equilibrium conditions (2.1) that $\mathrm{Def}_0{}^T\,\tau$ for any τ from $\Sigma^{(r)}$, meaning that also $\langle\tau, \mathrm{Def}^{(r)}\,\mathbf{u}\rangle = 0$ for $\mathbf{u} \in \Lambda'$, i.e., $\mathrm{Def}^{(r)}\,\mathbf{u}$ belongs to the set $\mathbf{E}^{(r)}$. In particular, if $\mathbf{e} = \mathrm{Def}\,\mathbf{u} \in \mathbf{E}$, then $\mathbf{e}^{(r)}$ belongs to the set $\mathbf{E}^{(r)}$. Using another obvious inequality $D^{(r)}(\mathbf{e}^{(r)}) \leqslant D(\mathbf{e})$ we find

$$\beta_l^{(r)} \leqslant \inf\{D^{(r)}(\mathbf{e}^{(r)})/\langle \mathbf{s}_l, \mathbf{e}^{(r)}\rangle : \mathbf{e} \in \mathbf{E},\ \langle s_l, \mathbf{e}\rangle > 0\} \leqslant \beta_l = \alpha_l$$

When taking account of the relations noted above $\alpha_l^{(r)} \leqslant \beta_l^{(r)}$, $\alpha_l^{(r)} = \alpha_l$ this results in the equality $\alpha_l^{(r)} = \beta_l^{(r)} = \alpha_l$. Finally, since the relationships $D(\mathbf{e}) = \langle\sigma, \mathbf{e}\rangle$, $\sigma \in C$ are satisfied for the extremals σ, $\mathbf{e}$ of Problem (2.3) and the stresses $S^{(r)}$ belongs to the space (by Assumption 3), then

$$\alpha_l = \frac{\langle\sigma, \mathbf{e}\rangle}{\langle \mathbf{s}_l, \mathbf{e}\rangle} = \frac{\langle\sigma, \mathbf{e}^{(r)}\rangle}{\langle \mathbf{s}_l, \mathbf{e}^{(r)}\rangle} \leqslant \frac{D^{(r)}(\mathbf{e}^{(r)})}{\langle \mathbf{s}_l, \mathbf{e}^{(r)}\rangle} \leqslant \frac{D(\mathbf{e})}{\langle \mathbf{s}_l, \mathbf{e}\rangle} = \alpha_l$$

which means the minimum of (2.5) is achieved on $\mathbf{e}^{(r)}$. Together with the previous remarks this results in the next assertion on the connection between problems (2.3) and (2.5).

Let Assumptions 1-3 be satisfied and let the set U be compact in the space $\mathbf{V}_0$. Then: 1) problems (2.5) are dual $\alpha_l^{(r)} = \beta_l^{(r)}$; 2) the extrema (2.5) agree with the load safety factor α_l; 3) the stress problems (2.3) and (2.5) agree; 4) if the extrema (2.3) are achieved on σ, $\mathbf{e} = \mathrm{Def}\,\mathbf{v}$, then the extrema (2.5) are reached on σ, $\mathbf{e}^{(r)}$.

This assertion shows how other expansions of the original formulation of limit load theory problems can be obtained from the problem (2.2) (among them in particular is the problem examined in /17/). They can be used to find the load safety factor. On the other hand, this assertion yields relations which the fields σ, $\mathbf{v}^{(r)}$ satisfy and from which conditions on discontinuity surfaces are derived in the long run.

Space Λ, $S^{(r)}$. We later consider the spaces

$$\Lambda = \Lambda_{p'}(\Omega, S_v) = \mathbf{L}_{p'}(\Omega) \times \mathbf{L}_\infty(S_q), \quad p' \geqslant 3$$

$$S^{(r)} = S_{p'}^{(r)}(\Omega, S_v) = \{\sigma \in S : \mathrm{Div}_0\,\sigma \in \mathbf{L}_{p'}(\Omega),\ \sigma_\mathbf{v}|_{S_q} \in \mathbf{L}_\infty(S_q)\}$$

$$(\mathrm{Div}_0\,\sigma)_{ij} = \partial\sigma_{ij}/\partial x^j$$

$$\|\sigma\|_{S^{(r)}}^2 = \|\sigma\|_S^2 + \|\mathrm{Div}_0\,\sigma\|_{\mathbf{L}_{p'}}^2 + \|\sigma_\mathbf{v}|_{S_q}\|_{\mathbf{L}_\infty}^2$$

The quantity $\sigma_\mathbf{v}|_{S_q}$ is determined by using mapping of the trace (Lemma 4.1 in /18/) as a linear continuous function on $\mathbf{W}_2^{1/2}(\partial\Omega)_3$ while $\sigma_\mathbf{v}|_{S_q}$ is its contraction in the set of such functions $\mathbf{w}$ from $\mathbf{W}_2{}^{1/2}(\partial\Omega)$ for which $\mathrm{supp}\,\mathbf{w} \subset S_q$.

The embedding of the space Λ into $\mathbf{V}_0'$ is given by the relation

$$\langle\mathbf{v}, \mathbf{l}\rangle = \int_\Omega \mathbf{f}\mathbf{v}\,dx + \int_{S_q} \mathbf{q}\mathbf{v}\,ds, \quad \mathbf{l} = (\mathbf{f}, \mathbf{q}) \in \Lambda, \quad \mathbf{v} \in \mathbf{V}_0$$

Here the right-hand side is defined since the space $\mathbf{V}_0$ is embedded continuously in $\mathbf{L}_p(\Omega)$, $p^{-1} + p'^{-1} = 1$, $1 \leqslant p \leqslant {}^3/_2$, and there is a continuous mapping of the trace from $\mathbf{V}_0$ into $\mathbf{L}_1(\partial\Omega)$ (see /8/, say). For the same reason the mentioned embedding is continuous.

Since the continuity of the embedding of $S^{(r)}$ into the space S is also obvious, Assumption 1 is satisfied. Conditions sufficient to satisfy Assumptions 2 and 3, and of course, also, to validate the assertion formulated above about the connectedness of the problems (2.3) and (2.5), are given by the following proposition.

Lemma 1. Let Ω be a bounded domain of the class C^1 and let the set U be compact in the space $\mathbf{V}_0$. Then: 1) if τ belongs to the space $S^{(r)}$ then $\mathrm{Def}_0{}^T\,\tau = (-\mathrm{Div}_0\,\tau, \tau_\mathbf{v}|_{S_q})$ and therefore it belongs to the space Λ; 2) if the load $\mathbf{l} = (\mathbf{f}, \mathbf{q})$ from Λ is equilibrated by a stress field $\tau \in S$, then $\mathrm{Div}_0\,\tau = -\mathbf{f}$, $\tau_\mathbf{v}|_{S_q} = \mathbf{q}$ and therefore, τ belongs to the space $S^{(r)}$.

The proof relies on Lemma 4.1 in /18/.

Piecewise-smooth fields σ, u *and the integral relation.* It is later assumed that the conditions under which there is the above-mentioned connectedness of problem (2.3) and (2.5), are satisfied. By virtue of this connectedness, the extremals σ, $\varepsilon_0 = \mathrm{Def}^{(r)}\mathbf{u}$ $(\mathbf{u} = \mathbf{v}^{(r)})$ of the problem (2.5) correspond to the fields σ, $\mathbf{v}$ for which there is the equality $m_s(\sigma) = M_k(\mathbf{v})$.

When $\sigma, \mathbf{u}$ are discontinuous piecewise-smooth fields, the required integral relation results from their extremal properties.

The velocity u as an element of the space Λ' is the pair $(\mathbf{u}_I, \mathbf{u}_F)$, $\mathbf{u}_I \in \mathbf{L}_p(\Omega)$, $\mathbf{u}_F \in \mathbf{L}_\infty'(S_q)$ /17/. We will say that the stress $\sigma \in S^{(r)}$ and the velocity $\mathbf{u} \in \Lambda'$ are piecewise-smooth if the domain Ω is divided into a finite number of regular domains ω_a $(a = 1, 2, \ldots, M)$,. for each of which $\sigma \in \mathbf{C}^1(\overline{\omega}_a)$, $\mathbf{u}_I \in \mathbf{C}^1(\partial\omega_a)$ and, moreover, $\mathbf{u}_F \in \mathbf{L}_1(S_q)$. Here the domain ω is called regular if the formula for integration by parts

$$\int\limits_{\omega} w \operatorname{div} \mathbf{v}\, dx = \int\limits_{\partial\omega} w \mathbf{v}_\nu\, dx - \int\limits_{\omega} \mathbf{v} \operatorname{grad} w\, dx$$

is valid for any functions $\mathbf{v}, w$ continuously differentiable on $\overline{\omega}$.

The domains ω_a $(a = 1, 2, \ldots, M)$ under consideration are called smoothness domains for the fields $\sigma, \mathbf{u}$. The smoothness domains for the piecewise-smooth fields $\sigma, \mathbf{u}$ can obviously be selected with considerable arbitrariness, for instance, by reduction.

The fields σ, $\mathbf{u}$ can suffer a discontinuity on the boundaries of the smoothness domains. The functions $\mathbf{u}_I$ are denoted by $[\mathbf{u}] = [\mathbf{u}_I]$ on the discontinuity surfaces. The external normal is selected on the boundary of the domain Ω and by definition we set $[\mathbf{u}] = \mathbf{u}_F - \mathbf{u}_I|_{S_q}$ on the surface S_q and $[\mathbf{u}] = -\mathbf{u}_I|_{S_v}$ on the surface S_v. The notation $\Gamma = \bigcup_a \partial\omega_a$ is also used.

Lemma 2. Let the extrema (2.5) be achieved on the stresses σ and the strain rates ε_0 where $\varepsilon_0 = \mathrm{Def}^{(r)}\mathbf{u}$, $\mathbf{u} \in \Lambda'$ (in particular $\varepsilon_0 = \mathbf{e}^{(r)}$, $\mathbf{u} = \mathbf{v}^{(r)}$, $\mathbf{e} = \mathrm{Def}\,\mathbf{v}$ and $m_s(\sigma) = M_k(\mathbf{v})$) can exist). Let the stress σ and the velocity u be piecewise-smooth. Then for any field $\sigma_* \in C^{(r)}$ that is continuously differentiable in the smoothness domains of the field σ, $\mathbf{u}$ the following inequality is satisfied

$$\int\limits_{\Gamma} (\sigma - \sigma_*)_{ij} [u_i] \nu_j\, ds + \int\limits_{\Omega \setminus \Gamma} (\sigma - \sigma_*)_{ij} \frac{\partial u_i}{\partial x^j}\, dx \geqslant 0 \tag{2.6}$$

Remark 1. The ordinary derivative $\partial \mathbf{u}_I/\partial x^j$, denoted by $\partial \mathbf{u}/\partial x^j$ in (2.6), is defined on $\Omega \setminus \Gamma$. The expression $\sigma_{ij}\nu_j$ has meaning on the surface Γ just as does $\sigma_{*ij}\nu_j)$ since the stresses σ belong to the space $S^{(r)}$ meaning, the condition $[\sigma_{ij}]\nu_j = 0$ is satisfied on the surface of discontinuity.

Proof. Since the extrema (2.5) agree (equal to the safety factor α_l), then for the corresponding extremals σ, $\varepsilon_0 = \mathrm{Def}^{(r)} u$ the following relationships are satisfied

$$\sigma = \sigma_0 + \alpha_l s_l, \quad \sigma_0 \in \Sigma^{(r)}, \quad \sigma \in C^{(r)}, \quad D^{(r)}(\varepsilon_0) = \alpha_l \langle s_l, \varepsilon_0 \rangle$$

The equality $\langle \sigma_0, \mathrm{Def}^{(r)}\mathbf{u} \rangle = 0$ is satisfied for the stresses $\sigma_0 \in \Sigma^{(r)}$ and the strain rates $\mathrm{Def}^{(r)}\mathbf{u}$ (see above). Consequently, the equality $\langle \sigma, \varepsilon_0 \rangle = D^{(r)}(\varepsilon_0)$ results from the preceding relationships. By definition this means that the inequality $\langle \sigma - \sigma_*, \varepsilon_0 \rangle \geqslant 0$, is satisfied for any $\sigma_* \in C^{(r)}$, and which with the equality $\varepsilon_0 = \mathrm{Def}^{(r)}\mathbf{u}$ and taking the first assertion of Lemma 1 into account is represented in the form

$$-\int\limits_{\Omega} \frac{\partial (\sigma - \sigma_*)_{ij}}{\partial x^j} u_{Ii}\, dx + \int\limits_{S_q} (\sigma - \sigma_*)_{ij} \nu_j u_{Fi}\, ds \geqslant 0$$

After integrating by parts in the first component over each of the smoothness domains of the fields $\sigma, \mathbf{u}$ and using the definition of $[\mathbf{u}]$, this inequality takes the form (2.6).

3. *Conditions on the discontinuity surfaces in an inhomogeneous rigid plastic body.*

We shall say that the properties of a body are continuous in the domain ω if the values $d(x, \mathbf{e})$ of the dissipation function for all points $x \in \omega$ and all tensors $\mathbf{e} \in \mathrm{Sym}^d$ are identical with the values of a certain function continuous in $\overline{\omega} \times \mathrm{Sym}^d$ (the bar denotes closure). The properties of a body filling the domain Ω, are called piecewise-continuous in Ω is separated into a finite number of domains in each of which the properties of the body are continuous.

We consider a sufficiently smooth common part γ of the domain boundary ω^+, ω^-, in each of which the properties of the body are continuous. Let d^+ (d^-) be a function continuous on $\bar{\omega}^+ \times \mathrm{Sym}^d$ (on $\bar{\omega}^- \times \mathrm{Sym}^d$) that is identical with the dissipation d on $\omega^+ \times \mathrm{Sym}^d$ (on $\omega^- \times \mathrm{Sym}^d$). Let $C_x^+ (C_x^-)$ be the set of allowable stresses defined for all points x from $\bar{\omega}^+$ (from $\bar{\omega}^-$) that corresponds to the dissipation d_x^+ (d_x^-). (We recall that there is a mutually one-to-one correspondence between C_x and d_x /4/).

The sets C_x^+, C_x^- of allowable stresses on both sides of the surface γ are distinct. In conformity with the definition presented above, the sets C are considered to be bodies with piecewise-continuous properties for which the stress field is allowable if it allowable in each of the continuity domains of the properties. In other words, composites are examined in which the bond is not weaker than the bounded parts.

Let $\mathbf{v}$ be the unit normal to the surface γ at the point x. We introduce the set B_x^+ (and B_x^- analogously)

$$B_x^+ = \{\boldsymbol{\tau} \in C_x^+ : \exists \boldsymbol{\tau}^- \in C_x^-,\quad \tau_{ij}\nu_j = \tau_{ij}^-\nu_j\}$$

We obtain conditions on the discontinuity surfaces for the stresses $\boldsymbol{\sigma}$ and the velocities $\mathbf{u}$ (which can, in particular, coincides with the discontinuity surfaces of the properties of the medium) in the form

$$\boldsymbol{\varepsilon}_\nu([\mathbf{u}]) \in N(\boldsymbol{\sigma}^+ \mid B_x^+),\quad \boldsymbol{\varepsilon}_\nu([\mathbf{u}]) \in N(\boldsymbol{\sigma}^- \mid B_x^-),\quad [\sigma_{ij}]\nu_j = 0 \qquad (3.1)$$
$$(\boldsymbol{\varepsilon}_\nu([\mathbf{u}]))_{ij} = ([u_i]\nu_j + [u_j]\nu_i)/2$$

Remark 2. The surfaces S_v and S_q are also later included among the discontinuity surfaces. The boundary condition $\mathbf{u}|_{S_v} = 0$ is firstly replaced thereby by the weaker condition $\mathbf{u}_I|_{S_v}\mathbf{v} = 0$ /19/ and, secondly, a velocity $\mathbf{u}_F$ different from the trace $\mathbf{u}_I|_{S_q}$ can be considered on the surface S_q /17/. We set $C_x^+ = \mathrm{Sym}$ and $\sigma_{ij}^+|_{S_q}\nu_j = q_i$ on the surfaces S_v and S_q in the conditions (3.1) according to the definition for their external sides with respect to the domain Ω. Then satisfaction of the relations (3.1) on S_v is equivalent to the condition $\boldsymbol{\varepsilon}_\nu([\mathbf{u}]) \in N(\sigma^- \mid C_x^-)$, and on S_q to the condition

$$\boldsymbol{\varepsilon}_\nu([\mathbf{u}]) \in N(\sigma^- \mid C_x^-),\quad \sigma_{ij}^-\nu_j|_{S_q} = q_i$$

Theorem. Let; 1) the properties of the body be piecewise-continuous; 2) the static and kinematic factors of the load $\mathbf{l}$ be equal for the stress field $\boldsymbol{\sigma} \in S^{(r)}$ and the velocity field $\mathbf{v} \in \mathbf{V}_0''$, $m_s(\boldsymbol{\sigma}) = M_k(\mathbf{v})$; 3) the stress field σ and velocity field $\mathbf{u} \equiv \mathbf{v}^{(r)}$ be piecewise smooth. Then the relations (3.1) are satisfied on the discontinuity surfaces of the fields $\boldsymbol{\sigma}, \mathbf{u}$ and the associated law (1.2) is satisfied in their smoothness domains.

For any interior point τ of the set B_{x_0}, $x_0 \in \Gamma$ an allowable piecewise-smooth stress field $\boldsymbol{\tau}_*$ is constructed in the neighbourhood of the point x_0, with a single discontinuity surface, the part of Γ lying on the neighbourhood under consideration, where $\boldsymbol{\tau}_*^+(x_0) = \tau, [\tau_{*ij}] = 0$. Furthermore, to prove relationships (3.1), the integral inequality (2.6) for $\sigma_* = \varphi\tau_* + (1-\varphi)\sigma$, should be used where φ is a smooth function $0 \leqslant \varphi \leqslant 1$, and the ordinary localization procedure is used by shrinking the support of the function φ in a suitable manner. Relationship (1.2) is proved analogously with appropriate simplifiation when the point x_0 lies in the smoothness domain of the fields σ, u.

Corollary 1. The condition $[u_i]\nu_i = 0$ is satisfied on the discontinuity surface. It results from the relationships (3.1) exactly as from (1.3) in /11/.

Corollary 2. In the special case of a homogeneous body the conditions (3.1) on the discontinuity surface reduce to the known relationships (1.3). This results from agreement of the sets $C_x^+ = C_x^- = C_x$, meaning also the set $B_x^+ = B_x^- = C_x$. Therefore, the conditions proposed in /10/ must be satisfied on the stress and velocity discontinuity surfaces if the latter results in equal static and kinematic load factors.

4. *Simple form and mechanical meaning of the conditions on the discontinuity surfaces.*

Using the unit normal $\mathbf{v}$ to the discontinuity surface, we set the scalar $P_\nu(\mathbf{s})$ and the vector $\mathbf{T}_\nu(\mathbf{s})$ with the components $\mathbf{T}_\nu{}^i$ in correspondence to each symmetric tensor of the second rank $\mathbf{s}$

$$P_\nu(\mathbf{s}) = s^{ij}\nu_i\nu_j,\quad T_\nu{}^i(\mathbf{s}) = s^{ik}\nu_k - s^{lm}\nu_l\nu_m\nu^i$$

The vector $\mathbf{T}_\nu(\mathbf{s})$ is obviously orthogonal to the vector $\mathbf{v}$. If $\boldsymbol{\sigma}$ is the stress tensor, then $P_\nu(\boldsymbol{\sigma})$ is the normal force, and $\mathbf{T}_\nu(\boldsymbol{\sigma})$ is the tangential force on an area with normal $\mathbf{v}$. When $\mathbf{a}\mathbf{v} = 0$ the inequalities

$$\boldsymbol{\varepsilon}_\nu(\mathbf{a})\cdot\mathbf{s} = 2\mathbf{T}_\nu(\boldsymbol{\varepsilon}_\nu(\mathbf{a}))\,\mathbf{T}_\nu(\mathbf{s}),\quad \mathbf{T}_\nu(\boldsymbol{\varepsilon}_\nu(\mathbf{a})) = \mathbf{a}/2,\quad P_\nu(\boldsymbol{\varepsilon}_\nu(\mathbf{a})) = 0 \qquad (4.1)$$

hold for the tensor $\varepsilon_\nu(\mathbf{a})$ with components $(\varepsilon_\nu(\mathbf{a}))_{ij} = (a_i\nu_j + a_j\nu_i)/2$.

Let $A_x^\pm$ be a set of vectors (orthogonal to the vector ν) in which the mapping of $\mathbf{T}_\nu$ goes over into the set $C_x^\pm$. Then the set $A_x^\circ = A_x^+ \cap A_x^-$ has the meaning of a set of tangential forces on the discontinuity surface that are allowable from the viewpoint of both conditions $\sigma^+(x) \in C_x^+$, $\sigma^-(x) \in C_x^-$. We note that the set A_x° is not generally identical with the set $\mathbf{T}_\nu(C_x^+ \cap C_x^-)$, but merely includes it.

The conditions (3.1) on the discontinuity surfaces are equivalent to the relations

$$[\mathbf{u}] \in N(\mathbf{t} \mid A_x^\circ), \quad \mathbf{t} \equiv \mathbf{T}_\nu(\sigma^+) = \mathbf{T}_\nu(\sigma^-), \quad P_\nu(\sigma^+) = P_\nu(\sigma^-) \qquad (4.2)$$
$$\sigma^+ \in C_x^+, \quad \sigma^- \in C_x^-$$

The equalities of the quantities $\mathbf{T}_\nu$ and P_ν are simply another way of writing the conditions $[\sigma_{ij}]\nu_j = 0$. To confirm the equivalence of the relationships (3.1) and (4.2) we use the equalities (4.1) and the possibility of representing any vector $\mathbf{t}_*$ from A_x° in the form $\mathbf{t}_* = \mathbf{T}_\nu(\sigma_*^+) = \mathbf{T}_\nu(\sigma_*^-)$ for certain σ_*^+ from C_x^+ and σ_*^- from C_x^-, where $P_\nu(\sigma_*^+) = P_\nu(\sigma_*^-)$.

The first of relations (4.2) has a geometric interpretation completely analogous to the interpretation of the associated law (1.2).

The sets A_x^+, A_x^-, A_x° that lie in the plane Π_ν orthogonal to the vector ν and the cones of the external normals to the set A_x° at the points K, L, M are shown in Fig.2 (compare with Fig.1). We note that even in the case of smooth flow surfaces (the boundaries of the sets C_x^+, C_x^-) the boundary of the set A_x° has angular points at the common position.

Finally, the first of the relationships (4.2) can be interpreted as the maximum principle

$$\mathbf{t} \in A_x^\circ, \quad (\mathbf{t} - \mathbf{t}_*)[\mathbf{u}] \geqslant 0 \quad \text{for any} \quad \mathbf{t}_* \in A_x^\circ \qquad (4.3)$$

It is exactly analogous to the well-known maximum principle $(\sigma - \sigma_*)\cdot \mathbf{e} \geqslant 0$, $\sigma_* \in C_x$, that connects the stress and strain rate in a smoothness domain. Here the tangential force plays the part of the stress while the velocity jump plays the part of the strain rate.

Therefore, conditions (3.1) or (4.2) on the discontinuity surface denote the following 1) the force on the discontinuity surface is continuous; 2) the stresses on both sides are allowable; 3) the velocity jump and the tangential force satisfy the law associated with the set of allowable tangential forces or, equivalently, the maximum principle (4.3).

Let us present certain simple properties of the conditions (4.2). It is assumed that the sets C_x^{d+}, C_x^{d-} are strictly convex (and, as always, contain the neighbourhood of zero). Then the set A_x° is also strictly convex.

1°. For any given direction the velocity jump $[\mathbf{u}]$, $[\mathbf{u}]\nu = 0$, the tangential force $\mathbf{t}$ is uniquely defined and lies on the boundary of at least one of the sets A_x^+, A_x^-.

2°. If the properties of the body undergo a discontinuity $(C_x^+ \neq C_x^-)$ then for a non-zero velocity jump $[\mathbf{u}]$ the stresses on one side of the discontinuity surface can lie strictly within the appropriate flow surface. The conditions (4.2) here remain arbitrary for their components.

3°. If the tangential force lies on the boundary of the set A_x^+, say, than the stresses σ^+ lie on the appropriate flow surface, the boundary of the set C_x^+. The deviator σ^{+d} is determined in a unique manner from the tangential force.

4°. In particular, if the tangential force is known and lies on the boundaries of both sets A_x^+, A_x^-, then both deviators σ^{+d} and σ^{-d} and the jumps of the global part are determined in a unique manner. Hence, the known condition of stress continuity results for a homogeneous body; it is not satisfied in the general case.

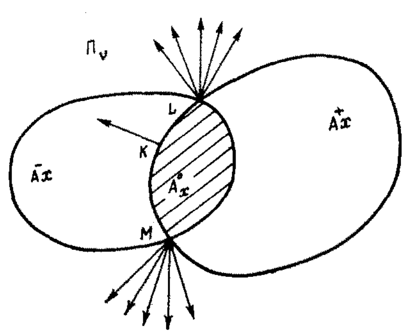

Fig.2

5. Decomposition of the velocity jump and the minimal dissipation property.

Let us mention still another useful representation of the relationships under consideration on discontinuity surfaces. All the vectors participating in the subsequent construction are orthogonal to the vector $\mathbf{v}$ or, speaking differently, lie in the plane Π_v.

The dissipation function δ_x° can be related to the set of allowable tangential forces A_x° introduced in the same way as for the set of allowable stresses. Namely, we set

$$\delta_x^\circ(\mathbf{w}) \quad \sup\{\mathbf{t}_*\mathbf{w} : \mathbf{t}_* \in A^\circ_x\} \tag{5.1}$$

for the vector $\mathbf{w}$ of the plane Π_v. The decomposition $[\mathbf{u}] = \mathbf{w}^+ + \mathbf{w}^-$ is examined for the velocity jump, where $\mathbf{w}^+, \mathbf{w}^-$ are vectors of the plane Π_v. The following assertions hold.

1°. The velocity jump $[\mathbf{u}]$ and the tangential force t satisfy the condition $[\mathbf{u}] \in N(\mathbf{t} \mid A_x^+)$ if and only if the decomposition

$$[\mathbf{u}] = \mathbf{W}^+ + \mathbf{W}^-, \quad \mathbf{W}^+ \in N(\mathbf{t} \mid A_x^+), \quad \mathbf{W}^- \in N(\mathbf{t} \mid A_x^-) \tag{5.2}$$

is possible.

2°. This decomposition possesses the following minimal property:

$$\min\{d_x^+(\varepsilon_v(\mathbf{w}^+)) + d_x^-(\varepsilon_v(\mathbf{w}^-)) : \mathbf{w}^+ \in \Pi_v,\ \mathbf{w}^- \in \Pi_v,\ \mathbf{w}^+ + \mathbf{w}^- = [\mathbf{u}]\} \tag{5.3}$$

is achieved on the vectors $\mathbf{w}^+ = \mathbf{W}^+$, $\mathbf{w}^- = \mathbf{W}^-$.

3°. The minimal value of (5.3) equals

$$d_x^+(\varepsilon_v(\mathbf{W}^+)) + d_x^-(\varepsilon_v(\mathbf{W}^-)) = \delta_x^\circ([\mathbf{u}]) = \mathbf{t}[\mathbf{u}]$$

For the sets A_x^+, A_x^- we introduce the dissipation functions δ_x^+, δ_x^- analogously to the relationship (5.1). The equalities

$$\delta_x^+(\mathbf{w}^+) = d_x(\varepsilon_v(\mathbf{w}^+)), \quad \delta_x^-(\mathbf{w}^-) = d_x^-(\varepsilon_v(\mathbf{w}^-))$$

follow from (4.1).

Then the problem (5.3) takes the form

$$\min\{\delta_x^+(\mathbf{w}^+) + \delta_x^-(\mathbf{w}^-) : \mathbf{w}^+ \in \Pi_v,\ \mathbf{w}^- \in \Pi_v,\ \mathbf{w}^+ + \mathbf{w}^- = [\mathbf{u}]\} \tag{5.4}$$

The necessary and sufficient conditions of the extremum /9/ for this problem are converted to the relationships (5.2) by the Marrow-Rockafellar theorem or to the relation $[\mathbf{u}] \in N(\mathbf{t} \mid A_x^\circ)$. This proves the first two assertions. The third assertion follows from the fact that the minimum in the problem (5.4) is reached according to Theorem 1 in Sect.3.4 from /9/ and equals

$$\delta_x^+(\mathbf{W}^+) + \delta_x^-(\mathbf{W}^-) = \delta_x^\circ([\mathbf{u}])$$

The equality $\delta_x^\circ([\mathbf{u}]) = \mathbf{t}[\mathbf{u}]$, in the third assertion follows for $\mathbf{t} \in A_x^\circ$ from the condition $[\mathbf{u}] \in N(\mathbf{t} \mid A_x^\circ)$, since this is equivalent to the maximum principle (4.3).

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ON ASYMPTOTIC INTEGRATION OF THE EQUATIONS OF MOTION OF MECHANICAL SYSTEMS SUBJECTED TO RAPIDLY OSCILLATING FORCES*

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An algorithm for the direct expansion of solutions of the Cauchy problem in a small parameter in a finite time interval is proposed in the development of the idea in the author's paper /1/ for systems of differential equations describing the motion of mechanical systems subjected to rapidly oscillating forces.

We consider a mechanical system whose motion is described by the vector differential equation

$$A(q)q^{\cdot\cdot} + B(q)q^{\cdot} = F(t, q) + \omega\Phi(t, q, \tau) \tag{1}$$

where $q = (q^1, \ldots, q^n)$ is the generalized coordinate vector, the dot denotes differentiation with respect to time t, A is a positive-definite matrix of the inertial forces, B is the matrix of the dissipative forces, $\omega\Phi$ are large amplitude oscillating forces ($\omega \gg 1$, $\tau = \omega t$). For simplicity we will consider Φ to be a trigonometric polynomial in τ of period 2π, with zero mean in τ. Let the following initial conditions be given

$$q(0) = \alpha,\ q^{\cdot}(0) = \beta \tag{2}$$

We will seek the approximate solution of the Cauchy problem (1) and (2) in the form

$$q^* = u_0(t) + \omega^{-1}[u_1(t) + v_1(t, \tau)] + \ldots + \omega^{-s}[u_s(t) + v_s(t, \tau)] + \ldots \tag{3}$$

where $v_i(t, \tau)$ are periodic functions of τ of period 2π with zero mean value. The sum $u_0 + \omega^{-1}u_1 + \ldots$ is the smooth motion component while $\omega^{-1}v_1 + \omega^{-2}v_2 + \ldots$ is the vibrational component. We have

$$A(q^*) = A^\circ + \omega^{-1}A_q^\circ(u_1 + v_1) +$$
$$\omega^{-2}A_q^\circ(u_2 + v_2) + {}^1/_2\overset{\circ}{A}_{qq}(u_1 + v_1)^2 + \cdots$$
$$(A^\circ = A(u_0),\ A_q^\circ = A_q(u_0), \ldots)$$

Analogous expressions hold for $B(q^*), F(t, q^*), \ldots$.

We obtain from the initial conditions (2), formulas (3) and the result of differentiating (3) with respect to t

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